

# Optimal Retirement Consumption with a Stochastic Force of Mortality<sup>☆</sup>

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## Abstract

We extend the lifecycle model (LCM) of consumption over a random horizon (a.k.a. the Yaari model) to a world in which (i.) the force of mortality obeys a diffusion process as opposed to being deterministic, and (ii.) a consumer can adapt their consumption strategy to new information about their mortality rate (a.k.a. health status) as it becomes available. In particular, we derive the optimal consumption rate and focus on the impact of mortality *rate* uncertainty vs. simple *lifetime* uncertainty – assuming the actuarial survival curves are initially identical – in the retirement phase where this risk plays a greater role.

In addition to deriving and numerically solving the PDE for the optimal consumption rate, our main general result is that when utility preferences are logarithmic the initial consumption rates are identical. But, in a CRRA framework in which the coefficient of relative risk aversion is greater (smaller) than one, the consumption rate is higher (lower) and a stochastic force of mortality does make a difference.

That said, numerical experiments indicate that even for non-logarithmic preferences, the stochastic mortality effect is relatively minor from the individual's perspective. Our results should be relevant to researchers interested in calibrating the lifecycle model as well as those who provide normative guidance (a.k.a. financial advice) to retirees.

**Keywords:** lifecycle consumption, stochastic mortality, survival curve matching, JEL codes: E21/G22, Subject Category: ??, Insurance Branch Category: ??

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## 1. Introduction and Motivation

The lifecycle model (LCM) of savings and consumption – originally postulated by Fisher (1930) and refined by Modigliani and Brumberg (1954), Modigliani (1986) – is at the core of most multi period asset pricing and allocation models, as well as the foundation of microeconomic consumer behavior. The original formulation – for example Ramsey (1928) and Phelps (1962) – assumed a deterministic horizon. But, in a seminal

contribution, the LCM was extended by Yaari (1964, 1965) to a stochastic lifetime, which eventually led to the models of Merton (1971), Richard (1975) and hundreds of subsequent papers on asset allocation over the human lifecycle.

The conceptual under-pinning of the LCM is the intuitive notion of *consumption smoothing* whereby (rational) individuals seek to minimize disruptions to their standard of living over their entire life. They plan a consumption profile that is continuous, equating marginal utility at all points, based on the assumption of a concave utility function. See the recent (and very accessible) article by Kotlikoff (2008) in which this concept is explained in a non-technical way.

Once again, until the seminal contribution by Yaari (1964, 1965), the LCM was employed by economists in an idealized world in which death occurred with probability one at some terminal horizon. Menahem Yaari introduced lifetime uncertainty into the lifecycle model, in addition to – his more widely known contribution of – introducing actuarial notes and annuities into optimal consumption theory.

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<sup>☆</sup>This research is supported in part by grants from NSERC, MITACS/Mprime and The IFID Centre. The authors acknowledge helpful comments from an IME reviewer, as well D. Babbel, T. Davidoff, G. Harrison, M. Lachance, S.F. Leung and from seminar participants at the Lifecycle, Insurance, Finance and Economics (LIFE) conference at the Fields Institute, Georgia State University, McMaster University, the Risk Theory Seminar (RTS) at Florida State University and the University of Technology, Sydney. A previous version of this paper circulated as: *Yaari's lifecycle model in the 21st century: consumption under a stochastic force of mortality*

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In the expressions (and theorems) he derived for the optimal consumption function, Yaari (1965) assumed a very general *force of mortality* for the remaining lifetime random variable, without specifying a particular law. His results would obviously include a constant force of mortality (i.e. exponential remaining lifetime) as well as Gompertz-Makeham (GM) mortality, and other commonly formulated approximations. Yaari provided a rigorous foundation for Irving Fisher's claim that lifetime uncertainty effectively increases consumption impatience and is akin to behavior under higher subjective discount rates. Mathematically, the mortality rate was added to the subjective discount rate.

That said, most of the empirical or prescriptive papers in the LCM literature have not gone beyond assuming the GM law – or some related deterministic function – for calibration purposes. In other words, mortality is just a substitute for subjective discount rates. In fact, one is hard-pressed to differentiate high levels of longevity and mortality risk aversion from weak preferences for consumption today vs. the future, i.e. patience. Some have labeled this risk neutrality with respect to lifetime uncertainty.

For example: Levhari and Mirman (1977), Davies (1981), Deaton (1991), Leung (1994), Butler (2001), Bodie et al. (2004), Dybvig and Liu (2005), Kingston and Thorp (2005), Babbel and Merrill (2006), Park (2006), Wallmeier and Zainhofer (2007), Feigenbaum (2008), or the recent work by Lachance (2012) – all assume a deterministic force of mortality.

Indeed, some economists continue (surprisingly) to ignore mortality altogether, for example the recent review by Attanasio and Weber (2010). Perhaps this is because when the force of mortality is deterministic, it can be added to the subjective discount rate without any impact on the mathematical structure of the problem.

To our knowledge, the only authors within the financial economics literature that have considered the possibility of non-constant mortality rates in a life-cycle model are Cocco and Gomes (2009), although their Lee-Carter mortality model is not quite *stochastic* as in Milevsky and Promislow (2001), Dahl (2004), Cairns et al. (2006), or the various models described in the book by Pitacco et al. (2008), or the concerns expressed by Norberg (2010).

Moreover, a number of very recent papers – for example Menoncin (2008), Stevens (2009) and Post (2010) – have examined the implications of (truly) stochastic mortality rates on the demand and pricing of certainly annuity products, but have not derived the impact of stochasticity on optimal consumption alone or examined the impact of pure uncertainty in the mortality rate.

Another related paper is Bommier and Villeneuve (2012) who examine the impact of relaxing the assumption of additively separable utility and what-they-call risk neutrality with respect to life duration. But, they also assume a deterministic force of mortality in their formulation and examples. In that sense, our work is similar because we also relax the so-called risk neutrality and the intertemporal additivity.

In sum, to our knowledge, none of the existing papers within the LCM literature have assumed a stochastic force of mortality – which is the model of choice in the current actuarial and insurance literature – and then derived its impact on pure consumption behavior. We believe this to be a foundational question, and in this paper our objective is straightforward, namely, to compare the impact of stochastic vs. deterministic mortality rates on the optimal consumption rate.

### 1.1. A Proper Comparison

Assume that two hypothetical retirees – i.e. consumers who are not expecting any future labour income – approach a financial economist for guidance on how they should spend their accumulated financial capital over their remaining lifetime; a time horizon they both acknowledge is stochastic. Assume both retirees have time-separable and rational preferences and seek to maximize discounted utility of lifetime consumption under the same elasticity of intertemporal substitution ( $1/\gamma$ ), the same subjective discount rate ( $\rho$ ) and the same initial financial capital constraint ( $F_0$ ). They have no declared bequest motives and – for whatever reason – neither are willing (or able) to invest in anything other than a risk-free asset with instantaneous return ( $r$ ); which means they are *not* looking for guidance on asset allocation or annuities.<sup>1</sup> All they want is an optimal consumption plan ( $c^*(t); t \geq 0$ ) guiding them from time zero (retirement) to the last possible time date of death ( $t \leq D$ ). Most importantly, both retirees agree they share the same probability-of-survival curve denoted by  $p(s)$ . In other words they currently live in the same health state and the same effective biological age. For example, they both agree on a  $p(35) = 5\%$  probability that either of them survive for 35 years and a  $p(20) = 50\%$  probability that either of them survive for 20 years, etc.

Yaari (1964, 1965) showed exactly how to solve such a problem. He derived the Euler-Lagrange equation for the optimal trajectory of wealth and the related consumption function.

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<sup>1</sup>This simplification is made purely to focus attention on the impact of stochastic mortality.

In Yaari's model both of the above-mentioned retirees would be told to follow identical consumption paths until their random date of death. In fact, they would both be guided to optimally consume  $c(t)^* = F(t)/a(t)$ , where  $a(t)$  is a *function of time only* and is related to an actuarial annuity factor. We will explain this factor in more detail, later in the paper.

But here is the impetus for our comparison. Although both retirees appear to have the same longevity risk assessment and agree on the survival probability curve  $p(s)$ , they have *differing views about the volatility of their health as proxied by a mortality rate volatility*. In the language of current actuarial science, the first retiree (1) believes that his instantaneous force of mortality (denoted by  $\lambda^{\text{DfM}}(t)$ ) will grow at a deterministic rate until he eventually dies, while the second retiree (2) believes that her force of mortality (denoted by  $\lambda^{\text{SfM}}(t)$ ) will grow at stochastic (but measurable) rate until a random date of death. As such, the remaining lifetime random variable for retiree 2 is doubly stochastic. While this distinction might sound farfetched and artificial, a growing number of researchers in the actuarial literature are moving to such models <sup>2</sup>, rather than the simplistic mortality models traditionally used by economists. The actuaries' motivation in advocating for a stochastic force of mortality, is to generate more robust pricing and reserving for mortality-contingent claims. These studies have all argued that SfM models better reflect the uncertainty inherent in demographic projections *vis a vis* the inability of insurance companies to diversify mortality risk entirely. We ask: *how do the recent actuarial models impact the individual economics of the problem?*

When one thinks about it, real-life mortality rates are indeed stochastic, capturing (unexpected) improvements in medical treatment, or (unexpected) epidemics, or even (unexpected) changes to the health status of an individual. Rational consumers choosing to make saving and consumption decisions using models based on deterministic mortality rates would likely agree to re-evaluate those decisions if their views about the values of those mortality rates change dramatically. Our thesis is that economic decision-making can only be improved if mortality models reflect the realistic evolution of mortality rates.

We will carefully explain the mathematical distinction between deterministic and stochastic forces of mortality (SfM) in Section 2 of this paper, but just to

make clear here, at time zero both our hypothetical retirees agree on the initial survival probability curve  $p(s)$ . However, at any future time their perceived survival probability curves will deviate from each other depending on the realization of the mortality rate between now and then.

Motivated by such models of mortality, in this paper we derive the optimal consumption function for both retirees; one who believes in – and operates under a – stochastic mortality and one who does not. Stated differently, we will solve the (consumption only) Yaari (1965) model where the optimal consumption plan is given as a function of wealth, time *and the evolving mortality rate as a state variable*. Indeed, with thousands of LCM papers in the economic literature over the last 50 years, and the growing interest in stochastic mortality models in the actuarial community, we believe these results will be of interest to both communities of researchers.

Recall that in the Yaari model conditioning on the mortality rate was redundant or unnecessary since its evolution over time was deterministic. All one needed was the value of wealth  $F(t)$  and time  $t$ . But, in a stochastic mortality model, the mortality rate itself becomes a state variable. In this paper we show how the uncertainty of mortality interacts with longevity risk aversion ( $\gamma$ ) – which is the reciprocal of the intertemporal elasticity of substitution – to yield an optimal consumption plan. Mortality no longer functions as just a discount rate.

To briefly preview our results, we describe the conditions under which retiree 1 (deterministic mortality) will start-off consuming more than retiree 2 (stochastic mortality), as well the conditions under which retiree 1 consumes less than retiree 2, and the (surprising) conditions under which they both consume exactly the same. We provide numerical examples under a variety of specific mortality models and examine the magnitude of this effect.

The remainder of this paper is organized as follows. In Section 2 we explain in more detail exactly how a stochastic model of mortality differs from the more traditional (and widely used in economics) deterministic force of mortality. In Section 3 we take the opportunity to review the (consumption only) Yaari (1965) model and set our notation and benchmark for the stochastic model. In Section 4 we characterize the optimal consumption function in the stochastic mortality model under the most general assumptions, and prove a theorem regarding the relationship between consumption in the two models. In Section 5 we make some specific assumptions regarding the stochastic mortality rate and illustrate the magnitude of this effect, and Section 6 sum-

<sup>2</sup>We appreciate and acknowledge comments made by a referee, that models in which mortality depends on health status, which itself is stochastic, have been used by actuaries well-before the introduction of 21st century stochastic mortality models.

marizes our main results and concludes the paper. The appendix contains mathematical details and algorithms that are not central to our main economic contributions.

First, we explain exactly the difference between deterministic and stochastic force of mortality.

## 2. Understanding The Force of Mortality

In most of the relevant papers in the LCM literature over the last 45 years the force of mortality from time zero to the last possible date of death is known with certainty. Ergo, the conditional survival probabilities over the entire retirement horizon are known (in advance) at time zero. So, if a 65-year-old retiree is told (by his doctor) that he faces a 5% chance of surviving to age 100 and a 37% chance of surviving to age 90, then by definition there is a  $13.5\% = (0.05/0.37)$  probability of surviving to age 100, if he is still alive at age 90. In other words, he makes consumption decisions today that trade-off utility in different states of nature, knowing that if-and-when he reaches the age of 90, there will only be a 13.5% chance he will survive to age 100. In the language of actuarial science, the table of individual  $\{q_{x+i}; i = 0, \dots, N\}$  mortality rates is known in advance. This is the essence of a deterministic force of mortality and textbook life contingencies. If  $q_{65}$  is the retiree's probability of dying between age 65 and 66, while  $q_{66}$  is the probability of the same retiree dying between age 66 and 67, then the probability of surviving from age 65 to age 67 is  $(1 - q_{65})(1 - q_{66})$ .

In stark contrast, under a stochastic force of mortality the above multiplicative relationship breaks down. We do not know in advance how survival probabilities will evolve. While a 65-year-old might currently face a 5% estimated probability of surviving to age 100 and a 37% chance of reaching age 90, there is absolutely no guarantee that the conditional survival probability from any future age, to age 100 (given the observed mortality rates), will satisfy the ratio. At time zero there is an expectation of what the probability will be at age 90. But, the probability itself is random. This way of thinking – which might be new to economists – is the essence of a stochastic force of mortality and is the impetus for our paper.

Here it is formally. Let  $\lambda(t)$  denote the mortality rate of a cohort of a population, which may be stochastic or deterministic. Let  $\mathcal{F}_t = \sigma\{\lambda(q) \mid q \leq t\}$  be the filtration determined by  $\lambda$ . Then individuals in the population have lifetimes of length  $\zeta$  satisfying

$$P(\zeta > s \mid \zeta > t, \mathcal{F}_\infty) = e^{-\int_t^s \lambda(q) dq}. \quad (1)$$

Assume further that  $\lambda(t)$  is a Markov process, and define the survival function  $p(t, s, \lambda)$  by

$$p(t, s, \lambda) = E\left[e^{-\int_t^s \lambda(q) dq} \mid \lambda(t) = \lambda\right]. \quad (2)$$

This gives the conditional probability of surviving from time  $t$  to time  $s$ , given knowledge of the mortality rate at time  $t$ . Therefore

$$\begin{aligned} P(\zeta > s \mid \zeta > t, \mathcal{F}_t) \\ = E\left[e^{-\int_t^s \lambda(q) dq} \mid \mathcal{F}_t\right] = p(t, s, \lambda(t)). \end{aligned} \quad (3)$$

If  $t = 0$  then we write  $p(s, \lambda)$  for  $p(0, s, \lambda)$ .

Our basic problem in this paper will be to compare optimal consumption under two models that share a common initial value  $\lambda_0$  of the mortality rate, as well as a common survival function  $p(t, \lambda_0)$ . Typically one will be deterministic and one stochastic. When we do actual computations, we will either choose a specific deterministic model and calibrate a stochastic model to it, or conversely, we will choose a stochastic model and calibrate the deterministic model to it. Both possibilities are discussed below. It should be clear from the context which model we are discussing. But when it is necessary to make this distinction explicitly, we will write  $\lambda^{\text{DfM}}(t)$  and  $\lambda^{\text{StM}}(t)$ .

### 2.1. Deterministic force of Mortality (DfM)

Let  $\lambda_0 = \lambda(0)$  be the initial value of the mortality rate. In the deterministic case,

$$p(t, \lambda_0) = e^{-\int_0^t \lambda(q) dq}, \quad (4)$$

and we can recover  $\lambda(t)$  as  $-p_t(t, \lambda_0)/p(t, \lambda_0)$ , where the  $t$ -subscript denotes the time derivative. In other words, if we start with a concrete stochastic model, and obtain the survival curve  $p(t, \lambda_0)$  from it, the above formula determines the calibration of the deterministic force of mortality model. This approach is computationally simpler, but has the disadvantage that neither the stochastic nor deterministic model is in a simple form, familiar to and used by practitioners. In other words, a “simple” model for the stochastic force of mortality rates leads to a “complicated” model for the deterministic force of mortality, and vice versa.

When doing actual calculations we will start by assuming that  $\lambda(t)$  follows a standard Gompertz model. The Gompertz model was introduced in 1825, but more recently was popularized by Carrière (1994), for example. Alternative models are presented in

Guterman and Vanderhoof (1998) and others are discussed as early as Brillinger (1961). In our case, we use:

$$d\lambda(t) = \eta\lambda(t) dt \quad (5)$$

so  $\lambda(t) = \lambda_0 e^{\eta t}$ . The usual form for Gompertz is  $\lambda(t) = b^{-1} e^{(x+t-m)/b}$ , so here we are using  $\eta = 1/b$  and  $\lambda_0 = b^{-1} e^{(x-m)/b}$ . This model is simple, and takes advantage of long experience calibrating the Gompertz model to real populations.

Note that in the deterministic setting,

$$\begin{aligned} p(t, s, \lambda(t)) &= e^{-\int_t^s \lambda(q) dq} \\ &= e^{-\int_0^s \lambda(q) dq} / e^{-\int_0^t \lambda(q) dq} = \frac{p(s, \lambda_0)}{p(t, \lambda_0)}. \end{aligned} \quad (6)$$

This will typically NOT be true in the stochastic setting. As long as we keep in mind that we are calibrating at time 0 (i.e. to  $p(t, \lambda_0)$  only) that should not cause problems.

Table 1 displays a typical (loosely based on U.S. unisex annuitant mortality) deterministic mortality survival probability “matrix” of values together with the corresponding mortality rate at each age  $x$ , on the bottom row. Note that these numbers were generated using a (deterministic) Gompertz model in which  $m = 89.335$  and  $b = 9.5$ . Indeed, given the initial probability of survival from age 65 to any age  $y > 65$  (which is the first column in Table 1) one can solve for the conditional survival probability from age  $y$  to any age  $z > y$ , by dividing the two probability values. This is the essence of equation (6). Alas, when mortality rates are stochastic all numbers  $p(t, s, \lambda(t))$  beyond the first column in Table 1, are unknown at time zero.

## 2.2. Stochastic Force of Mortality (SfM)

There are many possible stochastic models to choose from. Starting from the models of Lee and Carter (1992), Cairns et al. (2006) as well as Wills and Sherris (2010), actuaries have employed a variety of specifications for the stochastic  $\lambda(t)$ , subsequently used to price mortality and longevity risk. In what follows in the numerical examples, we adopt a lognormal mortality rate, which is often called the Dothan model for interest rates in the derivative pricing literature – see Dothan (1978). Although it might seem natural to have constant drift and diffusion coefficients, in order to calibrate to a given deterministic model, we allow a time-dependent growth coefficient. For most of the numerical examples provided later-on we take:

$$d\lambda(t) = \mu(t)\lambda(t) dt + \sigma\lambda(t) dB(t) \quad (7)$$

where  $B(t)$  is a Brownian motion. This is obviously the source of randomness in the stochastic force of mortality. There are many ways to select (or calibrate) a stochastic force of mortality to a particular survival curve. The details on how to actually compute this are provided in the second part of the appendix.

With the probability background out of the way, we now review the (consumption only) Yaari (1965) model which is based on a deterministic force of mortality.

## 3. Review of the Yaari (1965) Model

The canonical lifecycle model (LCM) with a random date of death and assuming no bequest motive, can be written as follows:

$$J = \max_c E \left[ \int_0^D e^{-\rho t} u(c(t)) 1_{\{t \leq \zeta\}} dt \right], \quad (8)$$

where  $\zeta$  is the remaining lifetime satisfying  $\Pr[\zeta > t] = p(t, \lambda_0)$ , defined above in Section 2. We fix a (deterministic) last possible time  $D$  of death, so  $\zeta \leq D$ . When the mortality rate is deterministic one can obviously assume independence between the optimal consumption  $c^*(t)$  and the lifetime indicator variable  $1_{\{t \leq \zeta\}}$ , so that by Fubini’s theorem we can re-write the value function as:

$$\begin{aligned} J &= \max_c \int_0^D e^{-\rho t} u(c(t)) E[1_{\{t \leq \zeta\}}] dt \\ &= \max_c \int_0^D e^{-\rho t} u(c(t)) p(t, \lambda_0) dt. \end{aligned} \quad (9)$$

From this perspective, there really is not any more randomness in the model. This is a problem within the calculus of variations subject to some constraints on the function  $c(t)$ . In the end, the survival probability is absorbed into the discount rate.

Let  $r$  denote the risk free interest rate. To avoid the distractions of inflation models and assumptions, throughout this paper we assume that  $r$  is expressed in real (after-inflation) terms and therefore consumption  $c(t)$  is expressed in real terms as well. The wealth (budget) constraint can then be written as:

$$F_t(t) = rF(t) + \pi_0 - c(t), \quad (10)$$

with boundary conditions  $F(0) = W > 0$  and  $F(D) = 0$ . We are using the subscript  $F_t$  to denote a first derivative w.r.t time, and if needed  $F_{tt}$  for the second derivative. The parameter  $\pi_0$  denotes a constant income rate which we include in this section for comparison with Yaari’s model, but which in subsequent sections will be taken

Table 1: Conditional Survival Probability: Deterministic Mortality								
	$x = 65$	$x = 70$	$x = 75$	$x = 80$	$x = 85$	$x = 90$	$x = 95$	$x = 100$
To Age 65	1.000							
To Age 70	0.9479	1.000						
To Age 75	0.8659	0.9135	1.000					
To Age 80	0.7429	0.7837	0.8580	1.000				
To Age 85	0.5733	0.6047	0.6620	0.7716	1.000			
To Age 90	0.3696	0.3899	0.4268	0.4975	0.6447	1.000		
To Age 95	0.1758	0.1855	0.2031	0.2367	0.3067	0.4757	1.000	
To Age 100	0.0500	0.0527	0.0577	0.0673	0.0872	0.1353	0.2844	1.000
$\lambda(x)$	0.0081	0.0137	0.0232	0.0394	0.0667	0.1129	0.1911	0.3234

to equal zero;  $c(t)$  is the consumption rate and the control variable in our problem. In a follow-up paper we hope to examine the impact of additional factors, such as different interest rates for borrowing versus lending, of the availability of actuarial notes (i.e. the case when the interest rate is  $r + \lambda(t)$ ).

In this paper we operate under a constant relative risk aversion (CRRA) formulation for the utility function. In principle this should mean using  $\bar{u}(c)$ , where:

$$\bar{u}(c) = \frac{c^{1-\gamma} - 1}{1-\gamma} \quad (11)$$

for  $\gamma > 0$  and  $\gamma \neq 1$ , with the understanding that when  $\gamma = 1$  we define  $\bar{u}(c) = \ln c$ . This family of utilities varies continuously with  $\gamma$ . The marginal utility of consumption is the derivative of utility with respect to  $c$ , which is simply

$$u_c = c^{-\gamma} > 0. \quad (12)$$

Of course, it makes no difference to our optimization problem (and the optimal control) if we shift  $\bar{u}$  by an arbitrary additive constant. So to make scaling relationships easier to express, actual calculations will be carried out using the equivalent utilities

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad (13)$$

for  $\gamma > 0$  and  $\gamma \neq 1$ . When  $\gamma = 1$  we take  $u(c) = \bar{u}(c) = \ln c$ .

As a consequence of the Euler-Lagrange Theorem, the optimal financial capital trajectory  $F(t)$  must satisfy the following linear second-order non-homogenous dif-

ferential equation over the values for which  $F(t) \neq 0$ .

$$F_{tt}(t) - \left( \frac{r - \rho - \lambda(t)}{\gamma} + r \right) F_t(t) + r \left( \frac{r - \rho - \lambda(t)}{\gamma} \right) F(t) = - \left( \frac{r - \rho - \lambda(t)}{\gamma} \right) \pi_0. \quad (14)$$

When the pension income rate  $\pi_0 = 0$  the differential equation collapses to the homogenous case. See Chiang (1992), for example for an exposition of Euler-Lagrange equations in economics.

### 3.1. Explicit Solution: Gompertz Mortality

When the (deterministic) mortality rate function obeys the (pure) Gompertz law of mortality

$$\lambda(t) = \frac{1}{b} \exp\left(\frac{x + t - m}{b}\right), \quad (15)$$

the survival probability can be expressed as

$$p(t, \lambda_0) = e^{-\int_0^t \lambda(q) dq} = e^{b\lambda_0(1-e^{t/b})}. \quad (16)$$

Here  $x$  denotes the age at time 0,  $m$  is called the modal value and  $b$  is the dispersion coefficient for the Gompertz model. To simplify notation let

$$k(t) = \frac{r - \rho - \lambda(t)}{\gamma}, \quad (17)$$

and recall from the budget constraint that:

$$c(t) = rF(t) - F_t(t) + \pi_0, \quad (18)$$

$$c_t(t) = rF_t(t) - F_{tt}(t). \quad (19)$$

Equation (14) can be rearranged as

$$F_{tt}(t) - rF_t(t) + k(t)(rF(t) - F_t(t)) = -k(t)\pi_0, \quad (20)$$

which then leads to

$$k(t)c^*(t) - c_t^*(t) = 0 \quad (21)$$

The solution to this basic equation is

$$\begin{aligned} c^*(t) &= c^*(0)e^{\int_0^t k(s)ds} = c^*(0)e^{\int_0^t \frac{r-\rho-\lambda(s)}{\gamma}ds} \\ &= c^*(0)e^{\frac{r-\rho}{\gamma}t} e^{-\frac{1}{\gamma}\int_0^t \lambda(s)ds} = c^*(0)e^{\frac{r-\rho}{\gamma}t} p(t, \lambda_0)^{1/\gamma}, \end{aligned} \quad (22)$$

where  $c^*(0)$  is the optimal initial consumption rate, to be determined, which is the one free constant resulting from equation (21). Note that when the interest rate  $r$  is equal to the subjective discount rate  $\rho$ , and  $\gamma = 1$  (i.e. log utility), the optimal consumption rate at any age  $x + t$  is the probability of survival to that age times the initial consumption  $c^*(0)$ . However, when  $\gamma > 1$ , which implies higher levels of risk aversion, the optimal consumption rate will decline at a slower rate as the retiree ages. Longevity risk aversion induces people to behave as if they were going to live longer than determined by the actuarial mortality rates. We will explore the impact of  $\gamma$  on the optimal consumption path in a stochastic force of mortality model, later in Section 4, which is why it's important to focus on this here.

Mathematically one can see that  $(p(t, \lambda_0))^{1/(\gamma+\varepsilon)}$  is greater than  $(p(t, \lambda_0))^{1/\gamma}$  for any  $\varepsilon > 0$  since  $p(t, \lambda_0) < 1$  for all  $t$ . Finally, note that in the Gompertz mortality model evaluating  $(p(t, \lambda_0))^{1/\gamma}$  for a given  $(x, m, b)$  triplet is equivalent to evaluating  $p(t, \lambda_0)$  under the same  $x, b$  values, but assuming that  $m^* = m + b \ln \gamma$ . This then implies that one can tilt/define a new deterministic mortality rate  $\lambda_0 = \gamma \lambda_0$  and derive the optimal consumption as if the individual was risk neutral. This will be used later in the explicit expression for  $F(t)$  and  $c^*(t)$ .

Moving on to a solution for  $F(t)$ , we now substitute the optimal consumption solution (22) into equation (18) to arrive at yet another first-order ODE, but this time for  $F(t)$ :

$$F_t(t) - rF(t) - \pi_0 + c^*(0)e^{\frac{r-\rho}{\gamma}t} p(t, \lambda_0)^{1/\gamma} = 0. \quad (23)$$

Writing down the canonical solution to this equation leads to:

$$\begin{aligned} F(t) &= e^{rt} \left( \pi_0 \int_0^t e^{-rs} ds \right. \\ &\quad \left. - c^*(0) \int_0^t e^{\frac{r-\rho}{\gamma}s} p(s, \lambda_0)^{1/\gamma} e^{-rs} ds + F(0) \right), \end{aligned} \quad (24)$$

where  $F_0$  denotes the free initial condition from the ODE for  $F(t)$  in equation (23). Recall that we still haven't specified  $c^*(0)$ , the initial consumption. We

will do so (eventually) by using the terminal condition  $F(D) = 0$ .

To represent the wealth trajectory explicitly define the following (new) Gompertz Present Value (GPV) function

$$\begin{aligned} a_x^T(r, m, b) &= \int_0^T p(s, \lambda_0) e^{-rs} ds = \int_0^T e^{-\int_0^s (r+\lambda(t))dt} ds \\ &= \int_0^T e^{-\int_0^s \left( r + \frac{1}{b} e^{\left( \frac{x-m+t}{b} \right)} \right) dt} ds \\ &= \frac{b\Gamma(-rb, \exp\{\frac{x-m}{b}\}) - b\Gamma(-rb, \exp\{\frac{x-m+T}{b}\})}{\exp\{(m-x)r - \exp\{\frac{x-m}{b}\}\}}. \end{aligned} \quad (25)$$

The function  $a_x^T(r, m, b) = a(t)$  is the age- $x$  cost of a life-contingent annuity that pays \$1 per year continuously provided the annuitant is still alive, but only until time  $t = T$ , which corresponds to age  $x + T$ . If the individual survives beyond age  $(x + T)$  the payout stops. Naturally, when  $T = \infty$  the expression collapses to a conventional single premium income annuity (SPIA).

Note that  $\Gamma(A, B)$  is the incomplete Gamma function. In other words, equation (25) is analytic and in closed-form.

The reason for introducing the GPV is that combining equation (24) with equation (25) leads to the (very tame looking) expression

$$F(t) = \left( F(0) + \frac{\pi}{r} \right) e^{rt} - a_x^T(r - k, m^*, b) c^*(0) e^{rt} - \frac{\pi_0}{r}, \quad (26)$$

where recall that  $m^* = m + b \ln \gamma$ . Then, using the boundary condition  $F_\tau = 0$ , where  $\tau$  is the wealth depletion time, we obtain an explicit expression for the initial consumption

$$c^*(0) = \frac{(F(0) + \pi_0/r) e^{r\tau} - \pi_0/r}{a_x^T(r - k, m^*, b) e^{r\tau}}. \quad (27)$$

### 3.2. Consumption Under DfM: Numerical Examples

In our numerical examples we assume an 86.6% probability that a 65-year-old will survive to the age of 75, a 57.3% probability of reaching 85, a 36.9% probability of reaching 90, a 17.6% probability of reaching age 95 and a 5% probability of reaching 100. These are the values generated by the Gompertz law with  $m = 89.335$  and  $b = 9.5$ . To complete the parameter specifications required for our model, we assume the subjective discount rate ( $\rho$ ) is equal to the risk-free rate  $r = 2.5\%$ . Within the context of a lifecycle model, this implies that the optimal consumption rates would be constant over time in the absence of longevity and mortality uncertainty.

We are now ready for some results. Assume a 65-year-old with a (standardized) \$100 nest egg. Initially we allow for no pension annuity income ( $\pi_0 = 0$ ) and therefore all consumption must be sourced to the investment portfolio which is earning a deterministic interest rate  $r = 2.5\%$ . The financial capital  $F(t)$  must be depleted at the very end of the lifecycle, which is time  $D = (120 - 65) = 55$  and there are no bequest motives. So, according to equation (27), the optimal consumption rate at retirement age 65 is \$4.605 when the risk aversion parameter is  $\gamma = 4$  and the optimal consumption rate is (higher) \$4.121 when the risk aversion parameter is set to (higher)  $\gamma = 8$ .

As the retiree ages ( $t > 0$ ) he/she rationally consumes less each year – in proportion to the survival probability adjusted for  $\gamma$ . For example, in our baseline  $\gamma = 4$  level of risk aversion, the optimal consumption rate drops from \$4.605 at age 65, to \$4.544 at age 70 (which is  $t = 5$ ), then \$4.442 at age 75 (which is  $t = 10$ ), then \$3.591 at age 90 (which is  $t = 25$ ) and \$2.177 at age 100 (which is  $t = 35$ ), assuming the retiree is still alive. A lower real interest rate ( $r$ ) leads to a reduced optimal consumption/spending rate. All of this can be sourced to equation (22).

Thus, one of the important insights is that a fully rational consumer will actually spend less as they progress through retirement. The optimizer spends more at earlier ages and reduces spending with age, even if his/her subjective discount rate (SDR) is equal to (or less than) the real interest rate in the economy.

Intuitively the individual deals with longevity risk by planning to reduce consumption – if that risk materializes – in proportion to the survival probability, linked to their risk aversion. The Yaari (1965) model provides a rigorous foundation to the statement by Fisher (1930) in his book *Theory of Interest* (page 85): “... *The shortness of life thus tends powerfully to increase the degree of impatience or rate of time preference beyond what it would otherwise be...*” and (page 90) “*Everyone at some time in his life doubtless changes his degree of impatience for income... When he gets a little older... he expects to die and he thinks: instead of piling up for the remote future, why shouldn't I enjoy myself during the few years that remain?*”<sup>3</sup>

<sup>3</sup>For additional (case specific) examples of the Yaari (1965) model in action during the non-labour income retirement phase, we refer the interested reader to Milevsky and Huang (2010) or a recent paper by Lachance (2012).

### 3.3. Time-zero Consumption Ratio = Initial Withdrawal Rate

Finally, in the very specific case when  $\pi_0 = 0$  (which implies that the wealth depletion time is  $\tau = D$ ) and the subjective discount rate  $\rho = r$ , the retiree must rely exhaustively on his/her initial wealth  $F_0$ . We get

$$\frac{c^*(0)}{F(0)} = \frac{1}{a_x^D(r - \lambda_0/\gamma, m^*, b)} \quad (28)$$

We now have all the ingredients to compare with a stochastic model. This ratio is often called the Initial Withdrawal Rate (IWR) amongst financial practitioners and in the retirement spending literature.

## 4. Optimal Consumption: General Results

In this section we obtain the *most general* optimal consumption strategy for a retiree maximizing expected discounted utility of consumption with uncertain lifetime, which will include the (consumption only) Yaari (1965) model as a special case. Since our main focus now is on the mortality model, at this stage we make the additional assumption  $\rho = r$ , that is, that the subjective discount rate equals the interest rate in the economy. Also, in contrast to the discussion in the previous section, we assume no exogenous pension income, so that  $\pi_0 = 0$ , which then precludes any borrowing. We now assume a fixed terminal horizon  $T$ , which denotes the last possible date of death. The mathematical formulation is to find

$$J = \max_{c(s) \text{ adapted}} E \left[ \int_0^T e^{-\int_0^s (r + \lambda(q)) dq} u(c(s)) ds \right] \quad (29)$$

$\lambda(0) = \lambda, F(0) = F$

Whereas in Section 3 of this paper we used calculus of variation techniques to derive the optimal trajectory of wealth and the consumption function, given the inclusion of mortality as a state variable we must resort to dynamic programming techniques to obtain the optimality conditions. Regardless of the different techniques, we will show how the optimal consumption function collapses to the Yaari (1965) model when the volatility of mortality is zero.

Define:

$$J(t, \lambda, F) = \max_{c(s) \text{ adapted}} E \left[ \int_t^T e^{-\int_t^s (r + \lambda(q)) dq} u(c(s)) ds \right] \quad (30)$$

$\lambda(t) = \lambda, F(t) = F$



As in the deterministic mortality model, the wealth process (which we shall soon see is stochastic) satisfies  $dF(t) = (rF(t) - c(t))dt$ . Assume that there is an optimal control. Then for that control,

$$\begin{aligned} E \left[ \int_0^T e^{-\int_t^s (r+\lambda(q)) dq} u(c(s)) ds \middle| \mathcal{F}_t \right] \\ = e^{-\int_0^t (r+\lambda(q)) dq} J(t, \lambda(t), F(t)) \\ + \int_0^t e^{-\int_t^s (r+\lambda(q)) dq} u(c(s)) ds \quad (31) \end{aligned}$$

is a martingale. This will likewise give a supermartingale under a general choice of  $c$ . Applying Itô's lemma, we obtain the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} \sup_c \{u(c) - cJ_F\} + J_t - (r + \lambda)J \\ + rFJ_F + \mu(t)\lambda J_\lambda + \frac{\sigma^2 \lambda^2}{2} J_{\lambda\lambda} = 0. \quad (32) \end{aligned}$$

If there is any possibility of confusion, we will denote this value function  $J^{\text{SfM}}(t, \lambda, F)$ .

For deterministic mortality, HJB can be obtained by sending  $\sigma \rightarrow 0$  with  $\mu(t) = \eta$ , which was equal to  $1/b$  in the Yaari (1965) model derived in Section 3, as

$$\sup_c \{u(c) - cJ_F\} + J_t - (r + \lambda)J + rFJ_F + \eta\lambda J_\lambda = 0. \quad (33)$$

Moving on to the optimal consumption plan, we solve the HJB equation under CRRA utility as follows: let

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad J = \frac{F^{1-\gamma}}{1-\gamma} a(t, \lambda), \quad (34)$$

where the second expression results from the scaling which follows from the first, and apply the first order condition  $c^* = J_F^{-1/\gamma}$ . We obtain  $c^* = Fa^{-1/\gamma}$  and get the following equation for  $a(t, \lambda)$ :

$$a_t - (r\gamma + \lambda)a + \gamma a^{1-1/\gamma} + \mu(t)\lambda a_\lambda + \frac{\sigma^2 \lambda^2}{2} a_{\lambda\lambda} = 0 \quad (35)$$

with boundary condition  $a(T, \lambda) = 0$ .

We now solve the PDE for  $a(t, \lambda)$ , which we re-write as:

$$\beta_t + 1 - \left(r + \frac{\lambda}{\gamma}\right)\beta + \mu(t)\lambda\beta_\lambda + \frac{\gamma-1}{2\beta}\sigma^2\lambda^2\beta_\lambda^2 + \frac{1}{2}\sigma^2\lambda^2\beta_{\lambda\lambda} = 0. \quad (36)$$

for  $\beta = \beta(t, \lambda) = a(t, \lambda)^{1/\gamma}$ . The boundary conditions are  $\beta(T, \lambda) = 0$ ,  $\beta_\lambda(t, \infty) = 0$  and at  $\lambda = 0$  we solve  $\beta_t + 1 - r\beta = 0$ . Note that the optimal consumption rate is  $c = F/\beta$ , using shorthand notation. On to the main theorem.

#### 4.1. Stochastic Force of Mortality: Main Theorem

Denote by  $c^{\text{SfM}}(t, \lambda, F)$  the optimal consumption at time  $t$ , given  $\lambda(t) = \lambda$  and  $F(t) = F$ , under a stochastic force of mortality (SfM) model. Denote by  $c^{\text{DfM}}(t, F)$  the optimal consumption at time  $t$ , when  $F(t) = F$ , under a deterministic force of mortality (DfM) model.

**Theorem 1.** Assume that the survival functions for the two models agree:  $p^{\text{SfM}}(t, \lambda_0) = p^{\text{DfM}}(t, \lambda_0)$  for every  $t \geq 0$ , and that utility is CRRA( $\gamma$ )

- (a)  $\gamma > 1 \implies c^{\text{SfM}}(0, \lambda_0, F) \geq c^{\text{DfM}}(0, F)$ ;
- (b)  $\gamma = 1 \implies c^{\text{SfM}}(0, \lambda_0, F) = c^{\text{DfM}}(0, F)$ ;
- (c)  $0 < \gamma < 1 \implies c^{\text{SfM}}(0, \lambda_0, F) \leq c^{\text{DfM}}(0, F)$ .

*Proof.* To see this, we change point of view, and work exclusively with the stochastic model. So we drop the SfM superscript, and write  $p = p^{\text{SfM}}$ ,  $J = J^{\text{SfM}}$ ,  $c^* = c^{\text{SfM}}$ ,  $\lambda = \lambda^{\text{SfM}}$ , etc. Within that model, we pose two different optimization problems, depending on the level of information available about  $\lambda(t)$ . The value function  $J(t, \lambda, F)$  solves the problem given before in (30), where  $c(t)$  can be any suitable process adapted to  $\mathcal{F}_t$ . But we define a new value function  $J^1(t, F)$  in which we impose an additional constraint on  $c(t)$ , namely that it be deterministic. More precisely,

$$J(0, \lambda_0, F_0) = \max_{c(s) \text{ adapted}} E \left[ \int_0^T e^{-\int_0^s (r+\lambda(q)) dq} u(c(s)) ds \right] \quad (37)$$

and

$$\begin{aligned} J^1(0, F_0) \\ = \max_{c(s) \text{ deterministic}} E \left[ \int_0^T e^{-\int_0^s (r+\lambda(q)) dq} u(c(s)) ds \right] \\ = \max_{c(s) \text{ deterministic}} \int_0^T e^{-rs} p(s, \lambda_0) u(c(s)) ds. \end{aligned} \quad (38)$$

We let  $c^*$  denote the optimal control for  $J$ , and  $c^1$  denote the optimal control for  $J^1$ .

Since every deterministic control  $c(t)$  is also adapted, we have the basic relationship

$$J(0, \lambda_0, F_0) \geq J^1(0, F_0). \quad (39)$$

On the other hand, the above expression is exactly what the old deterministic model would have given. That is,

$$J^1(0, F_0) = J^{\text{DfM}}(0, F_0) \quad (40)$$

and  $c^1 = c^{\text{DfM}}$ .

Due to scaling,  $J(t, \lambda, F) = a(t, \lambda)F^{1-\gamma}/(1-\gamma)$  and  $c^*(t, \lambda, F) = a(t, \lambda)^{-1/\gamma}F$  for some function  $a \geq 0$ . Likewise  $J^{\text{DfM}}(t, F) = a_1(t)F^{1-\gamma}/(1-\gamma)$  and  $c^1 = a_1^{-1/\gamma}F$  for

some  $a_1 \geq 0$ . If  $\gamma > 1$  then  $1 - \gamma < 0$ , so  $a(0, \lambda_0) \leq a_1(0)$ , so  $c^* \geq c^1$  at  $t = 0$ . This shows (a). Likewise if  $0 < \gamma < 1$  then  $a(0, \lambda_0) \geq a_1(0)$ , so  $c^* \leq c^1$  at  $t = 0$ . This shows (c).

Recall that when  $\gamma = 1$ , we have  $u(c) = \ln c$ . Earlier, when  $\gamma \neq 1$ , we had  $u(c) = c^{1-\gamma}/(1-\gamma)$  and could make use of a scaling relation. In other words, if  $c$  is optimal for  $F$ , then  $kc$  is optimal for  $kF$ , and that leads to the expression  $J(t, \lambda, kF) = k^{1-\gamma}J(t, \lambda, F)$ . Or in other words,

$$J(t, \lambda, F) = F^{1-\gamma}J(t, \lambda, 1). \quad (41)$$

With logarithmic utility, the corresponding expression is that

$J(t, \lambda, kF) = J(t, \lambda, F) + (\ln k) \int_t^T e^{-r(s-t)} p(t, s, \lambda) ds$ . Or in other words,

$$J(t, \lambda, F) = J(t, \lambda, 1) + (\ln F) \int_t^T e^{-r(s-t)} p(t, s, \lambda) ds. \quad (42)$$

Likewise,

$$J^{\text{DfM}}(t, F) = J^{\text{DfM}}(t, 1) + (\ln F) \int_t^T e^{-r(s-t)} \frac{p(s, \lambda_0)}{p(t, \lambda_0)} ds. \quad (43)$$

The first order conditions in the optimization problem then imply that

$$\begin{aligned} c^* &= F / \int_t^T e^{-r(s-t)} p(t, s, \lambda) ds, \\ c^{\text{DfM}} &= F / \int_t^T e^{-r(s-t)} \frac{p(s, \lambda_0)}{p(t, \lambda_0)} ds. \end{aligned} \quad (44)$$

These agree when we send  $t \rightarrow 0$ , showing (b).  $\square$

The theorem certainly proves that  $\gamma = 1$  is a point of indifference. The invariance of mortality volatility when utility is logarithmic is reminiscent of similar results in consumption theory where income negates substitution effects. More on this later.

Note that we only use  $J^1(0, F)$  above, not  $J^1(t, F)$ . If we had, we would have had to be careful. The correct definition is that

$$\begin{aligned} J^1(t, F) &= J^{\text{DfM}}(t, F) \\ &= \max_{c(s)} \int_t^T e^{-r(s-t)} \frac{p(s, \lambda_0)}{p(t, \lambda_0)} u(c(s)) ds \end{aligned} \quad (45)$$

rather than

$$\begin{aligned} \max_{c(s)} E \left[ \int_t^T e^{-\int_t^s (r+\lambda(q)) dq} u(c(s)) ds \right] \\ = \max_{c(s)} \int_t^T e^{-r(s-t)} E[p(t, s, \lambda(t))] u(c(s)) ds. \end{aligned} \quad (46)$$

These quantities have connections to annuities, as suggested by the fact that the optimal consumption rates given above are, as a fraction of wealth, inverse annuity prices. In particular,  $\int_t^T e^{-rs} p(s, \lambda_0) ds$  is the (actuarial) price of a deferred annuity, purchased at time 0 with payments starting at time  $t$ . While  $\int_t^T e^{-rs} E[p(t, s, \lambda(t))] ds$  is a forward annuity price. That is, if at time 0 an insurance company guarantees (a retiree) the right to buy an annuity at time  $t$  at a price determined at time 0, then this is that price (computed actuarially, i.e. by discounting mean cash flows).

#### 4.2. Intuition and Relation to Known Results

How should one interpret our result? It is tempting to view stochastic mortality as simply “more risky” than deterministic mortality, but that is not in fact the reason consumption shifts. The true explanation for our result is that the comparison can be reinterpreted equivalently as one between two different control problems, both within the context of the stochastic hazard rate model. Namely, a control problem in which the hazard rate is observed, so one can react to changes, versus one in which the hazard rate is not observed, so the control must be determined in advance. The utility in the deterministic model is the same as the utility for the second control problem (and indeed, this is the basis of our proof). So the mere presence of stochastic hazard rates will not cause a change in consumption; what shifts consumption is the ability to react to those changes.

There are two possible reactions to that ability to adjust consumption. One is to shift consumption into the future, taking advantage of the ability to adjust consumption upwards later, if the hazard rate should climb more than expected. The other reaction is to opt to consume more now, in the knowledge that one can cut back later if it seems likely that one will live longer than expected. Our message is that either reaction can be rational, and that which one is adopted depends on the person’s risk aversion, with the switch occurring at the point of logarithmic utility. The choice is between acting more conservatively in view of the possibility one might live longer, versus acting more aggressively in view of one’s ability to react to changes in the hazard rate.

There are other results in the literature where logarithmic utility is a qualitative point of indifference in behavior. An example – in a completely different context – is the classical result on the equilibrium pricing of assets derived by Lucas (1978)<sup>4</sup>. In a Lucas-type model

<sup>4</sup>We thank Thomas Davidoff for pointing out this analogy

– under logarithmic utility preferences – the equilibrium price of trees (or any other income producing asset for that matter) does not depend on the projected level of fruit output from those trees. The economic reason for this is that there are two effects on the current equilibrium price, of an increase in the expected future amount of fruit from trees. The first is the fact that at any given marginal utility of consumption of the fruit, the higher expected fruit production increases the attractiveness of owning trees today, which raises the current price of a tree. But, at the same time, the increased expected fruit output in future periods means higher consumption and lower marginal utility of consumption in that future period. This effect tends to reduce the attractiveness of owning trees today – the tree is going to pay off more in a time when marginal utility is expected to be low – and thus lowers the current value (and hence price) of a tree. These two forces are the manifestation of the (pure) income effect and substitution effect from the theory of *consumer choice*, and their net result – i.e. which actual dominates – depends on the shape (and curvature) of the utility function.

In the case of logarithmic utility, income and substitution effects are of the same size and opposite sign so the two forces exactly offset each other, leaving the current price of a tree unchanged in the face of a rise in expected future fruit output. This is (one of) the results from Lucas (1978). Although it does not appear the same powers are at force in our stochastic mortality model, this does illustrate that there are a number of settings in which one finds that logarithmic utility ( $\gamma = 1$ ) is the point of indifference between two opposing consumption effects.

## 5. Optimal Consumption: Numerical Examples

We started with a particular survival probability at time zero, namely the Gompertz mortality curve with parameters  $m = 89.335$  and  $b = 9.5$ . The age  $x = 65$  survival probabilities to any age  $y > x$  are given in Table 1. Both hypothetical retirees agree on these numbers, which means that their initial mortality rate is  $\lambda_0 = (1/9.5) \exp\{(65 - 89.335)/9.5\} = 0.008125$ .

Over time retiree 1 believes his mortality rate will grow at a rate  $\eta = (1/9.5) = 0.10526316$  per year, while retiree 2 believes it will evolve stochastically with a time-dependent growth rate of  $\mu(t)$  and a volatility

$\sigma$ . The actual curve  $\mu(t)$  depends on the selected parameter for volatility, since  $\mu(t)$  is constrained to match  $p(0, \lambda_0)$ . The actual process for extracting  $\mu(t)$  for any given value of  $\sigma$  is rather complicated (although it is not central to our analysis) and is placed in the appendix of this paper. With these numbers in hand – and specifically the function  $\mu(t)$  for the drift of the mortality rate – we can proceed to solve the PDEs given in equation (35) and (36), which then lead to the desired optimal consumption function and the initial portfolio withdrawal rate at age 65.

Table 2 provides a variety of numerical examples across different values of (mortality volatility)  $\sigma$  and (risk aversion)  $\gamma$ , once again assuming that the retirees are both at age  $x = 65$  with observable mortality rate  $\lambda_0 = 0.0081$ . As we proved in Section 3, and discussed above, the consumption rate is the same across all levels of mortality volatility when  $\gamma = 1$ . It increases relative to DfM when  $\gamma > 1$  and decreases relative to DfM when  $\gamma < 1$ . Notice the impact of stochastic mortality on optimal withdrawal rates is reduced as the value of risk aversion increases. Notice how at a coefficient of relative risk aversion  $\gamma = 10$ , the portfolio withdrawal rates are approximately 4.6% at all listed volatility levels.

Note that the  $\sigma$  values provided are rather *ad hoc* and have not been estimated from any particular demographic dataset. We refer the interested reader to recent actuarial papers – such as Bauer et al. (2008) – for an empirical discussion around the estimation of the volatility of mortality. Our objective here is to explore whether or not mortality volatility has a (noticeable) impact on rational behavior as opposed to on insurance pricing.

To sum up, when the coefficient of CRRA (denoted by  $\gamma$ ) is equal to one, and the retiree has logarithmic utility preferences, the optimal consumption rate at time zero is identical in both models. In other words, a retiree who cannot adjust their consumption plan as mortality rates evolve starts-off with the exact same consumption rate as the (more knowledgeable) consumer who can adapt to changes in mortality rates and health status. Although the path of their respective consumption will diverge over time – depending on the evolution of mortality rates – initially they are the same. In contrast, when the coefficient of CRRA is greater than one and the retiree is more risk-averse compared to a logarithmic utility maximizer, the initial consumption rate is higher in the stochastic model vs. the deterministic model. In other words, as one might expect the ability to adapt to changes in health status and new information about mortality rates allows the retiree to be more generous at time zero. Finally, when the coefficient of the CRRA is

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to the authors, and we refer the interested reader to the lecture notes by Christopher Carroll, available at the following website: [econ.jhu.edu/people/ccarroll/public/lecturenotes](http://econ.jhu.edu/people/ccarroll/public/lecturenotes), for this particular interpretation (and derivation) of the Lucas model.

<b>Table 2: Optimal Retirement Portfolio Withdrawal Rates <math>c^*(0)/F_0</math></b>						
<b>Mortality Volatility</b>	$\gamma = 0.5$	$\gamma = 1.0$	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 5$	$\gamma = 10$
$\sigma = 0$	7.59%	6.12%	5.58%	5.02%	4.78%	4.61%
$\sigma = 15\%$	7.52%	6.12%	5.60%	5.04%	4.80%	4.62%
$\sigma = 25\%$	7.44%	6.12%	5.62%	5.06%	4.82%	4.63%
Notes: Retirement age 65, interest rate $r = 2\%$ , mortality $\lambda_0 = 0.0081$						

between one and zero, the result is reversed. The canonical retiree in a stochastic mortality model will consume less compared to their neighbor who is operating under deterministic mortality assumptions.

Notwithstanding the above results, the absolute consumption rate at time zero is uniformly higher the lower the coefficient of relative risk aversion. This is a manifestation of longevity risk aversion. The retiree is concerned about living a long time, and therefore consumes less today to protect themselves and self-insure consumption in old age.

## 6. Discussion and Conclusion

In this article we extended the lifecycle model (LCM) of consumption over a random-length lifecycle, to a model in which individuals can adapt behavior to new information about mortality rates. The lifecycle model of saving and consumption continues to be very popular as a foundation model for decision-making amongst financial advisors, as recently described in the monograph by Bodie et al. (2008).

Yaari (1964, 1965) was the first to include lifetime uncertainty in a Ramsey-Modigliani lifecycle model and amongst other results, he provided a rigorous foundation for Irving Fisher's claim that lifetime uncertainty increases consumption impatience and is akin to higher subjective discount rates. When the mortality rate itself is stochastic, this analogy is no longer meaningful and – to our knowledge – the pure lifecycle model has not been extended into the realm of 21st century models of mortality and longevity risk.

We built this extension by assuming that (i) the instantaneous force of mortality is stochastic and obeys a diffusion process as opposed to being deterministic, and (ii) that a utility-maximizing consumer can adapt their consumption strategy to new information about their mortality rate (a.k.a. current health status) as it becomes available. Our diffusion model for the stochastic force of mortality was quite general, but inspired by (a.k.a. borrowed from) the recent literature in actuarial science. We focused our modeling attention on the

retirement income stage of the LCM where health considerations are likely to be more prevalent and to avoid complications induced by wages, labor and human capital consideration.

In the first part of this paper we re-derived the optimal consumption function under a deterministic force of mortality (DfM) using techniques from the calculus of variations. We provided a closed-form expression for the entire consumption rate function under a Gompertz mortality assumption. With those benchmark results in place, we derived the optimal consumption strategy under a stochastic force of mortality (SfM), by expressing and solving the relevant Hamilton-Jacobi-Bellman (HJB) equation. In addition to the time variable, two state variables in the resulting PDE are current wealth and the current mortality rate.

Retirees with (i) no bequest motives, (ii) constant relative risk aversion (CRRA) preferences, and (iii) subjective discount rates equal to the interest rate are expected to consume less as they age since they prefer to allocate consumption into states of nature where they are most likely to be alive. This is the conventional diminishing marginal utility argument. In our model, a positive shock to the mortality rate in the form of pleasant health news (perhaps a cure for cancer) will reduce consumption instantaneously and further than expected at time zero. A negative shock to the mortality rate (for example, being diagnosed with terminal cancer) will increase consumption beyond what was expected.

Moving forward, a natural extension would be to explore the impact of stochastic investment returns as well as mortality rates and include a strategic asset allocation dimension, *a la* Merton (1971). Another item on our research agenda is to explore the optimal allocation to health and mortality-contingent claims in a stochastic mortality model. Recall that one of the noted results of Yaari (1965) is that lifecycle consumers with no bequest motives should hold all of their wealth in actuarial notes. However, in the presence of a stochastic mortality, it is no longer clear how an insurance company would price pension annuities, given the systematic risk involved. In such a model, a retiree would have to choose between

investing wealth in a tontine pool, with corresponding stochastic returns or purchasing a pension annuity with a deterministic consumption flow, but possibly paying a risk-premium for the privilege. We conjecture that in a stochastic mortality framework, the optimal *product allocation* is a mixture of participating tontines and guaranteed annuities.

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## Appendix A. Matching Time-Zero Survival Curves

The calibration of our economic model leads to an interesting by-product problem in actuarial finance. In

particular, in order to construct a stochastic force of mortality that matches or fits a pre-determined Gompertz survival curve – the most popular and frequently used analytic law in this literature – one requires a log-normal diffusion process in which the drift itself grows even faster than exponentially over time. In this appendix we explain the mechanics of the procedure.

Given a deterministic model (Gompertz in our numerical examples), we compute the time-zero survival function  $p(t, \lambda_0)$ . We match this using a stochastic model, by a suitable choice of parameters. This means that at time 0 the two models deliver identical survival probabilities. Recall that at times other than  $t = 0$  the comparison will no longer be meaningful, even controlling for the current observed mortality rate, because the mismatch between conditional survival probabilities means that the two models give different views of life-times going forward.

Let  $\Lambda(t) = e^{-\int_0^t \lambda(q) dq}$  and define a pseudo-density  $q(t, \lambda)$  by the formula

$$E[\Lambda(t)\phi(\lambda(t))] = \int_0^\infty \phi(\lambda)q(t, \lambda) d\lambda. \quad (\text{A.1})$$

Then  $p(t, \lambda_0) = \int_0^\infty q(t, \lambda) d\lambda$ . By Itô's lemma,

$$\begin{aligned} \phi(\lambda(t))\Lambda(t) &= \phi(\lambda_0) + \int_0^t \Lambda(s) \left[ \mu(s)\lambda(s)\phi'(\lambda(s)) \right. \\ &\quad \left. + \frac{\sigma^2}{2} \lambda(s)^2 \phi''(\lambda(s)) - \lambda(s)\phi(\lambda(s)) \right] ds \\ &\quad + \int_0^t \Lambda(s) \mu(s) \lambda(s) \phi'(\lambda(s)) dB(s). \end{aligned} \quad (\text{A.2})$$

Take expectations and differentiate with respect to  $t$ . We get

$$\begin{aligned} \int_0^\infty \phi(\lambda)q_t(t, \lambda) d\lambda &= \int_0^\infty \left[ \mu(t)\lambda\phi'(\lambda) \right. \\ &\quad \left. + \frac{\sigma^2}{2} \lambda^2 \phi''(\lambda) - \lambda\phi(\lambda) \right] q(t, \lambda) d\lambda \end{aligned} \quad (\text{A.3})$$

with initial condition  $q(0, \cdot) = \delta_{\lambda_0}$ . Using integration by parts (for  $\phi$  vanishing fast at 0 and  $\infty$ ), we have

$$\begin{aligned} q_t(t, \lambda) &= -\mu(t) \frac{\partial}{\partial \lambda} [\lambda q(t, \lambda)] \\ &\quad + \frac{\sigma^2}{2} \frac{\partial^2}{\partial \lambda^2} [\lambda^2 q(t, \lambda)] - \lambda q(t, \lambda). \end{aligned} \quad (\text{A.4})$$

So if  $\mu(t)$  is known for  $0 \leq t \leq t_1$ , then all expectations  $\int_0^\infty q(t_1, \lambda)\phi(\lambda) d\lambda$  can be found by solving the forward equation for  $q$  and then integrating against  $\phi$ .

Let  $\lambda_{(1)} = \int_0^\infty \lambda q d\lambda$  and  $\lambda_{(2)} = \int_0^\infty \lambda^2 q d\lambda$  be the first two moments of  $q(t, \lambda)$ . Note that the zeroth moment is the survival probability, so we can integrate (by parts) the forward PDE for  $q$  and the product of  $\lambda$  and the forward PDE and obtain the following relationships

$$\begin{aligned} \lambda_{(1)} &= -\frac{dp}{dt}, \\ \lambda_{(2)} &= \mu(t)\lambda_{(1)} - \frac{d\lambda_{(1)}}{dt}. \end{aligned} \quad (\text{A.5})$$

Combined the two expressions, we have

$$\mu(t) = \frac{\frac{d\lambda_{(1)}}{dt} + \lambda_{(2)}}{\lambda_{(1)}}. \quad (\text{A.6})$$

Replacing  $\mu(t)$  in the forward PDE for  $q$  and obtain an integro-differential equation

$$\begin{aligned} q_t(t, \lambda) &= -\frac{\frac{d\lambda_{(1)}}{dt} + \lambda_{(2)}}{\lambda_{(1)}} \frac{\partial}{\partial \lambda} [\lambda q(t, \lambda)] \\ &\quad + \frac{\sigma^2}{2} \frac{\partial^2}{\partial \lambda^2} [\lambda^2 q(t, \lambda)] - \lambda q(t, \lambda), \end{aligned} \quad (\text{A.7})$$

or

$$\begin{aligned} q_t(t, \lambda) &= \frac{-\frac{\partial^2 p}{\partial t^2} + \int_0^\infty \lambda^2 q(t, \lambda) d\lambda}{\frac{\partial p}{\partial t}} \frac{\partial}{\partial \lambda} [\lambda q(t, \lambda)] \\ &\quad + \frac{\sigma^2}{2} \frac{\partial^2}{\partial \lambda^2} [\lambda^2 q(t, \lambda)] - \lambda q(t, \lambda), \end{aligned} \quad (\text{A.8})$$

which we can solve numerically with the initial condition  $q(0, \lambda) = \delta(\lambda - \lambda_0)$ .

We solve the integro-differential equation for  $q$  numerically first, obtain the value of  $\mu(t)$ . We then solve the HJB equation for optimal consumption as before, with the constant  $\mu$  now replaced by the function  $\mu(t)$  at  $\lambda = 0$ .

Finally, we should record a couple of remarks about the form of  $\mu(t)$ . First of all,

$$\mu(0) = \eta. \quad (\text{A.9})$$

To see this, observe that  $p_t(0, \lambda_0) = -E[\lambda_0 \Lambda_0] = -\lambda_0$ . So  $\lambda_0^2 = E[\lambda_0^2 \Lambda_0^2] = p_{tt}(0, \lambda_0) + \lambda_0 \mu(0)$ . But  $p_{tt}(0, \lambda_0)$  can be computed explicitly, since it is Gompertz, to give  $\lambda_0^2 - \lambda_0 \eta$ . This implies that  $\mu(0) = \eta$ .

Second, note that  $\mu(t)$  should be increasing in  $\sigma$ . The mean  $E[e^{-\int_0^t \lambda(q) dq}]$  does not change with  $\sigma$ , so by convexity of the exponential, the median of this quantity must decrease as we increase the variance. In other words,  $\mu(t)$  must rise. Put another way, this expectation is driven by the possibility of relatively larger values of the exponent, ie of abnormally low values of  $\lambda$ .

As  $\sigma$  rises, the impact of longevity risk gets more pronounced, and to compensate for that the growth rate  $\mu(t)$  must also rise.